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Quantum perturbation theory for the level splitting in billiards

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A perturbation theory is developed for the level splitting due to dynamical tunneling in two-dimensional billiards. Using a scattering-matrix approach, the splittings are expressed in terms of a matrix element connecting quasimodes localized in the subspace of positive and negative angular momentum, in analogy to the familiar degenerate double-well problem. The theory is shown to work well for billiards which are integrable, mixed, and strongly chaotic. [S1063-651X(98)51101-4]

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The eigenvalue spectra of quantum systems frequently exhibit doublets of quasidegenerate levels due to tunneling between classically noncommunicating regions of phase space. Of particular interest is the case when the regions are not separated by a potential energy barrier but by dynamical barriers due to local conservation laws in the classical motion. The "dynamical tunneling" [1] through such dynamical barriers has been the subject of a number of recent theoretical studies [2–7]. In contrast to the familiar potential tunneling, dynamical tunneling was found to be extremely sensitive to details of the classical motion. In particular, a dramatic enhancement of the level splittings was observed due to the presence of chaotic motion in classically inaccessible regions of phase space. Moreover, these regions induced strong fluctuations of the level splittings under variation of some external parameter. No simple scheme has been developed for calculating these splittings for a general problem, although detailed results have been obtained in the case of the annular billiard [6].

In this work we present an alternative approach for calculating the level splittings for a large class of two-dimensional billiards. The method describes not only the case of conventional dynamical tunneling, but also the case where quasidoublets arise from quantum dynamical localization of states [8]. Our method is based on the S-matrix formulation [9,10] to the quantization of billiards. We introduce a decomposition $S=S_0+V$, where S_0 defines an unperturbed problem with exact degeneracy between symmetry-related quasimodes localized in angular momentum. The tunnel splittings of the quasimodes are then calculated using perturbation theory. Using this method we find that dynamical tunneling in billiards can be formulated in close formal correspondence with the familiar double-well problem. A motivation for this work was recent studies indicating that both chaos-assisted tunneling and dynamical localization play a role in determining the linewidth of optical resonators of deformed circular shape [11,12].

We demonstrate the method by calculating the tunneling splittings for three classes of two-dimensional billiards: The elliptic billiard, the annular billiard [4,6], and "rough" billiards [8]. In both the elliptic billiard and the annular billiard quasidoublets are supported by regions of regular motion in phase space. However, while the elliptic billiard is integrable, the annular billiard is mixed, giving rise to a strong enhancement of tunneling ("chaos-assisted" tunneling) [4,6]. In contrast, the rough billiards studied here are almost completely chaotic and quasidoublets only occur due to quantum localization. For all three billiards we show that our results agree well with the exact splittings calculated numerically and that the method can obtain correct parameter dependences and statistical behavior.

In was first shown in Refs. [9,10] that the bound state spectrum of a billiard can be determined exactly from knowledge of the scattering matrix for waves incident on the billiard from outside ("inside-outside duality"). Here, we recall briefly the derivation [10] of the S matrix for billiards created by convex deformations of a circular boundary. Such billiards include the elliptic and rough billiards to be treated below. The boundary is parametrized in terms of the distance $R(\phi)$ between the origin and a point on the boundary in the ϕ direction. The wave function inside the billiard is expanded in term of cylindrical Hankel functions

$$\psi(r,\phi)=\sum_{n=-\infty}^{\infty} i^n[\alpha_n H_n^{(2)}(kr)+\beta_n H_n^{(1)}(kr)]e^{in\phi}. \quad (1)$$

The regularity of the wave functions at $r=0$ requires $\boldsymbol{\alpha}=\boldsymbol{\beta}$. The boundary condition $\psi(R(\phi),\phi)=0$ results in the second relation $\boldsymbol{\beta}=S^T(k)\boldsymbol{\alpha}$. Here the matrix $S(k)$ is given by $S(k)=-h^{(2)}(k)[h^{(1)}(k)]^{-1}$, where the matrices elements of $h^{(1,2)}(k)$ are defined by $h_{mn}^{(1,2)}=(1/2\pi)\int_0^{2\pi}d\phi H_m^{(1,2)}(kR(\phi))\exp[i(m-n)\phi]$. Both conditions together imply $\det[1-S(k)]=0$ at an eigenstate of the billiard, so that the zeros of the eigenphases of S determine the spectrum. Degenerate eigenphases will lead to degenerate levels, and small eigenphase splittings can be related to level splittings [6].

It can be shown that S as defined here *is* the S matrix for the scattering of angular momentum states impinging from outside the billiard [10], hence S is unitary. Further constraints on S are imposed by discrete symmetries of the billiard. The time-reversal symmetry $\psi=\psi^*$ requires $S_{m,n}=S_{-n,-m}$. Parity invariance for a billiard with the reflection symmetry $R(\phi)=R(-\phi)$ yields $S_{m,n}=S_{-m,-n}$. Time reversal and parity invariance together imply that S is symmetric.

A greyscale plot of $|S_{m,n}|$ for a typical rough billiard of the type defined in Ref. [8] is shown in Fig. 1. We used $R(\phi)=R_0+\Delta R(\phi)$ with $\Delta R(\phi)/R_0=\Sigma_{m=2}^{M}(\gamma_m/$

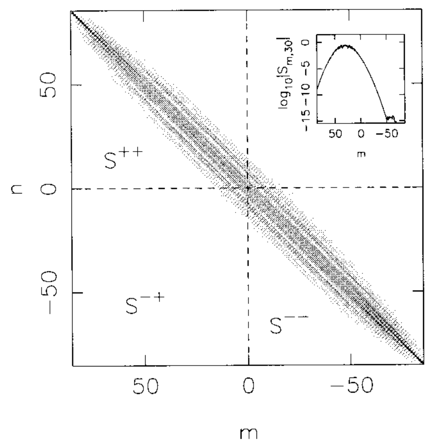

FIG. 1. Greyscale plot of $|S_{m,n}|$ for a rough billiard with $M=5$, $kR_0=80$, and $\kappa=0.045$. The dashed lines indicate the block division of S used in the perturbation theory. The inset shows the modulus of $S_{m,30}$.

$m)\cos(m\phi)$ where γ_m are real random coefficients. We chose $M=5$ and $\kappa=0.045$, where the average roughness κ is defined by $\kappa^2=\langle(dR/d\phi)^2/R_0^2\rangle_\phi$. Angular momenta $m,n>kR_0$, correspond semiclassically to states with impact parameters larger than the billiard radius, hence they only couple evanescently and S becomes almost diagonal for $m,n>kR_0$. Figure 1 reveals that scattering transitions between large positive and large negative angular momenta are strongly suppressed.

The perturbation theory is based on the observation that it is these transitions which are responsible for the tunnel splittings. That is, we can decompose the S matrix into two parts

$$S=S_0+V, \tag{2}$$

where S_0 maintains the sign of angular momentum upon scattering and S_0 has doubly degenerate eigenphases. In the angular momentum basis, S_0 and V are defined by the block decomposition

$$S=\begin{bmatrix} s^{++} & s^{+-} \\ s^{-+} & s^{--}\end{bmatrix},\quad S_0=\begin{bmatrix} s^{++} & 0 \\ 0 & s^{--}\end{bmatrix},$$

$$V=\begin{bmatrix} 0 & s^{+-} \\ s^{-+} & 0\end{bmatrix}. \tag{3}$$

By definition, only the perturbation V changes the sign of the angular momentum.

First we confirm that S has doubly degenerate eigenphases if V is neglected. From time-reversal symmetry $s^{++}=\mathcal{O}(s^{--})^T\mathcal{O}$, i.e., s^{++} and $(s^{--})^T$ are connected by a similarity transformation by the orthogonal matrix $\mathcal{O}_{m,n}=\delta_{m,-n}$. Since similar matrices have the same spectrum, and transposition preserves that spectrum, the matrices s^{++}, $(s^{--})^T$, and s^{--} have degenerate eigenphases. An analogous argument holds for billiards with only reflection symmetry.

The doublets of S_0 will be split due to the perturbation V. We emphasize that *in general* V is not a small operator. In fact, as shown in Fig. 1, the matrix elements $V_{m,n}$ with small $|m|$, $|n|$ are comparable in magnitude to the matrix elements of S_0. Therefore, V will be strongly mixing and the perturbation theory will fail badly for eigenvectors centered at small angular momentum. As expected, such eigenvectors do not give rise to quasidoublets. On the other hand, the matrix elements of V between eigenvectors of S_0 localized around large $|m|$ are quite small, indicating that the splittings of these vectors may be calculated perturbatively in V. It is crucial that S_0 does not describe the undeformed (circular) billiard (for which S and S_0 is diagonal), but takes into account most of the angular momentum mixing that occurs due to deformation. Only the small residual mixing that gives rise to splitting is neglected.

To find the leading order splittings of the eigenphases of S we apply degenerate perturbation theory. Since S_0 is not symmetric in the general case, we must distinguish between its left and right eigenvectors. Let $|R^+\rangle$ be a right eigenvector of S_0 such that $|R^+\rangle$ has nonzero components for positive angular momentum only. Due to time-reversal invariance, $\langle L^-|=\langle R^+|\mathcal{O}$ is a left eigenvector with the same unperturbed eigenvalue but with nonvanishing components for negative angular momenta only. Degenerate with this pair and defined in a similar way are eigenvectors $|R^-\rangle$, $\langle L^+|$. The eigenvalue shifts due to the perturbation are the roots of the 2×2 determinant

$$\begin{pmatrix} \langle L^+|V|R^+\rangle & \langle L^+|V|R^-\rangle \\ \langle L^-|V|R^+\rangle & \langle L^-|V|R^-\rangle \end{pmatrix}. \tag{4}$$

By definition $\langle L^+|V|R^+\rangle=\langle L^-|V|R^-\rangle=0$. Using time-reversal invariance this yields

$$\delta\theta=2|\langle L^+|V\mathcal{O}|L^+\rangle\langle R^+|\mathcal{O}V|R^+\rangle|^{1/2} \tag{5}$$

for the splitting of the eigenphases of S. For a billiard with time reversal and reflection symmetry, $|R^+\rangle=(\langle L^+|)^T$ and the result simplifies to

$$\delta\theta=2|\langle R^+|V\mathcal{O}|R^+\rangle|. \tag{6}$$

While Eqs. (5) and (6) give the splitting to first order in the perturbation V, higher-order corrections can be calculated applying the standard Rayleigh-Schrödinger perturbation theory. We found that the higher-order corrections can generally be neglected for the applications of our formalism to be presented below.

(i) *Elliptic billiard*: We have studied the level splittings for elliptic billiards with fixed area πR_0^2 and eccentricity e. To each quasidoublet we assign numbers (r,m), where r is the radial quantum number and $\pm m$ the angular quantum numbers of that doublet in the circular limit. To connect the eigenphase splitting with the splitting in k, we used the relation $\delta k_{r,m}\approx|\partial\theta_{r,m}^{(0)}/\partial k_{r,m}|^{-1}\delta\theta_{r,m}$, where $\theta_{r,m}^{(0)}$ is the eigenphase for the circle and we use $k_{r,m}\approx k_{r,m}^{(0)}$ as in Ref. [6].

For the ellipse, the eigenvectors needed for evaluating Eq. (6) are known explicitly [10]. In the angular momentum representation, their components are given by coefficients defining solutions of Mathieu's equation. Utilizing this mapping we obtained [13] the eigenvector components by a simple recursion relation. This yields analytical results in the regime of small deformations, $(ekR_0)^2\ll 1$, with $R_{m\pm 2p}^+\sim(ekR_0)^{2p}$

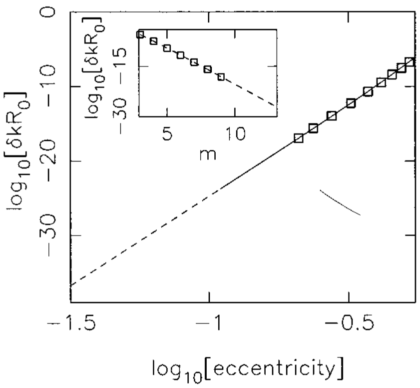

FIG. 2. Double-logarithmic plot of the level splitting as a function of the eccentricity e for the state $(r,m)=(0,12)$, calculated numerically (squares) and using the perturbation theory [solid line, dashed line in the regime $(ekR_0)^2 \ll 1$]. There is no numerical data for splittings less 10^{-18} due to the finite precision of our diagonalization routine. Errors for the eigenphase splitting itself are typically less than a few percent for $e<0.3$. Inset: $\log_{10}[\delta kR_0]$ for the states $(0,m)$ as a function of m for fixed eccentricity $e=0.1$.

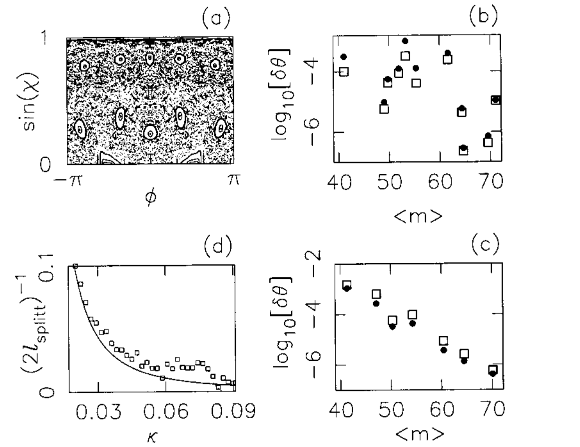

FIG. 3. (a) Classical phase space of the rough billiard, same parameters as in Fig. 1. χ is the angle of incidence with respect to the normal for trajectories scattering at the boundary. (b) Eigenphase splittings for $kR_0=80$ obtained numerically (squares) and using the perturbation theory (solid circles). (c) Mean eigenphase splittings averaged as described in the text. (d) Inverse tunneling length $(2l_{\rm splitt})^{-1}$ (squares) and inverse localization length $l^{-1}=0.25(\kappa kR_0)^{-2}$ (solid line) plotted vs κ.

for an eigenvector peaked at angular momentum m. We computed the perturbation V for small eccentricity by expanding S in powers of e^2. This showed $V_{p,-q}\sim e^{2(p+q)}$ for $p,q>0$. Substituting these results into Eq. (6) one finds that the splittings should increase with e as the power law $\delta k_{r,m}\sim e^{2m}$. Such a power-law dependence is expected for the integrable elliptical billiard [14]; however, note that one finds [13] exactly the same power-law splitting for the nonintegrable quadrupole billiard [11], although with a substantially larger prefactor.

In Fig. 2 we present a numerical test of our results for the state $(r,m)=(0,12)$. Shown is a double-logarithmic plot of $\delta k_{r,m}R_0$ as a function of e. The results clearly confirm the power law behavior with the exponent predicted by perturbation theory. Figure 2 (inset) shows that the level splittings $\delta k_{r,m}R_0$ for fixed $r=0$, $e=0.1$ decrease exponentially with m.

(ii) *Annular billiard:* The annular billiard was proposed and numerically investigated by Bohigas *et al.* [4] as a model system for chaos-assisted tunneling. This billiard has a circular boundary and a nonconcentric circular inclusion. Classically, angular momentum is exactly conserved and the motion is regular for all orbits that only scatter at the boundary, while orbits that hit the inner circle typically perform chaotic motion. Based on an S-matrix approach, Doron and Frischat [6] obtained analytical expressions for the tunnel splittings as a function of the S-matrix elements. These expressions were interpreted as arising from preferred tunneling paths in angular momentum space. Our goal in studying this billiard was to see if the same terms were selected by our simple first-order perturbation theory in V.

The S matrix of the annular billiard is not symmetric but has the reflection symmetry $S_{m,n}=S_{-m,-n}$ [6]. For the perturbation theory, this implies $|R^-\rangle=\mathcal{O}|R^+\rangle$, $|L^-\rangle=\mathcal{O}|L^+\rangle$. Using the determinant (4), this yields $\delta\theta=2|\langle L^+|V|R^-\rangle|$. We use the block-matrix model, which was introduced in Ref. [6], and argued to be statistically equivalent to the true S matrix of the problem. In the model the S matrix consists of regular blocks corresponding to angular momenta m far away from a chaotic region, edge blocks on the border, and a chaotic block. Within this block model the right eigenvectors of S peaked at angular momentum $-m$ are given by [6] $R^-_{-m}\approx 1$, $R^-_{-l}\approx S_{-l,-m}/d_{m,l}$, and $R^-_{\gamma}\approx\Sigma_l[S_{\gamma,-l}S_{-l,-m}]/[d_{m,l}d_{m,\gamma}]+S_{\gamma,-m}/d_{m,\gamma}$, respectively. Here, $-m$ labels a component in the regular block, indices $-l$, $l>0$ are used for the edge block and γ for the chaotic block; and $d_{m,l}=S_{m,m}-S_{l,l}$ and $d_{m,\gamma}=S_{m,m}-S_{\gamma,\gamma}$. The components of $\langle L^+|$ are given by similar equations as $|R^-\rangle$ but with $S_{-l,-m}$, $S_{\gamma,-m}$, and $S_{\gamma,-l}$ replaced by $S_{m,l}$, $S_{m,\gamma}$, and $S_{l,\gamma}$, respectively. Writing the eigenphase doublets in the form $\exp[i\theta_m\pm(1/2)\delta\theta_m]$ and expanding in $\delta\theta_m$, one obtains the eigenphase splitting as a sum over paths in index space. Among the various contributions is the term

$$\delta\theta_m^{(recer)}=2\,\mathrm{Im}\left(e^{-i\theta_m}\sum_{\gamma,l,l'}\frac{S_{m,l}S_{l,\gamma}S_{\gamma,-l'}S_{-l',-m}}{d_{m,\gamma}d_{m,l}d_{m,l'}}\right) \quad (7)$$

due to paths that lead from regular (r) states m to $-m$ through the edge (e) and chaotic (c) blocks. This is exactly the contribution due to the chaos-assisted paths that was found in Refs. [6]. Using the perturbation theory, we also find other paths of first order in V [e.g., paths of the type $(recr)$] that require further study.

(iii) *Rough billiards:* A Poincaré surface of section for a typical rough billiard with $M=5$ and average roughness $\kappa=0.045$ is shown in Fig. 3(a). It is characterized by small islands of regular motion together with a large chaotic region extending through most of phase space. This billiard is fundamentally different from the two studied above. There is no dynamical barrier to prevent classical transitions between positive and negative angular momenta. Therefore there is no semiclassical basis for the existence of quasidoublets. Here quasidoublets arise from the dynamical localization of the quantum wave functions [8].

We calculated the S matrix of a rough billiard for $kR_0=80$ and solved numerically for its eigenphases and eigenvectors. We compared the exact splittings with the prediction of perturbation theory calculated using the numerically determined eigenvectors. For each quasidoublet the eigenvectors were found to be peaked around some angular momentum $\langle \pm m \rangle$ [15]. Figure 3(b) displays strong fluctuations of the splittings vs $\langle m \rangle$. By varying parameters (kR_0, κ) we checked that these fluctuations result from avoided crossings as in previous [4–6] studies of chaos-assisted tunneling. The perturbation theory reproduces the fluctuations fairly well confirming that they derive from the structure of the unperturbed eigenvectors. To extract a systematic dependence on $\langle m \rangle$, we first averaged $\log_{10}[\delta\theta]$ over 50 values obtained by varying kR_0 in the interval [79,81]. Then, the average splittings were grouped into bins of $\Delta\langle m \rangle=4$, and mean values were obtained within each bin. Figure 3(c) shows that $\langle \log_{10}[\delta\theta] \rangle$ decreases approximately linearly with $\langle m \rangle$, $\langle \ln[\delta\theta] \rangle \approx \ln A - \langle m \rangle / l_{\rm splitt}$, where A is a constant. It turns out that the inverse slope $l_{\rm splitt}$ is related to the localization length l. In fact, assuming that the eigenvector components decay exponentially away from the peak, $a_m \sim \exp(-|m-\langle m \rangle|/l)$ for $m>0$, our perturbation theory predicts [13] $l_{\rm splitt}=l/2$. We checked this relation by calculating $l_{\rm splitt}$ for various values of κ and by comparing with the prediction [8] $l \approx 4(\kappa k R_0)^2$ for rough billiards. Good agreement is found [Fig. 3(d)] with no fitting parameters.

In summary, we have reduced the problem of finding tunnel splittings in billiards to that of evaluating the eigenvectors of S_0. In many cases it may be possible to either evaluate these eigenvectors approximately or model their statistical properties and hence obtain information about the splittings. The method is not semiclassical and works even when there is no classical dynamical barrier giving rise to doublets.

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